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Mean-field analysis of the equilibrium of patterns growing in a Laplacian field

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Abstract. The properties of the shape of patterns generated by random growth in a Laplacian field are investigated using a simple model recently introduced, the smoothed Laplacian model. The relation between this model and more complex systems such as Saffman–Taylor fingers or diffusion-limited aggregates, stands in the recently recognized universality of patterns whose evolution is controlled by the Laplace equation, and in an analytical approximation for the Laplacian field itself. The smoothed Laplacian model allows to treat analytically the equilibrium of the structure in the 'smoothed' Laplacian field in a mean-field approximation. The results on this model are extended in this paper to radial geometry thus providing theoretical insight in the dependence of the shape on the geometry of boundary conditions in which the process takes place and on the parameter that tunes the strength of the Laplacian field in the same way as the parameter η does for the dielectric-breakdown model. This new picture is compared to numerical simulation's results of DLA and DBM and proves to be consistent with them.

1. Introduction

The morphology of structures generated by processes ruled by the Laplace equation has attracted much interest in recent years. The close relation between the shape of Saffman–Taylor [1] fingers and the geometric properties of patterns produced by diffusion-limited aggregation (DLA) [2] or the dielectric-breakdown model (DBM) [3] has often been conjectured [1]. Apart from the fractal properties of DLA it has been recognized recently [4] that the distribution of occupation of the sites of the lattice in DLA follows the same profile of Saffman–Taylor fingers.

The universality of the shape of patterns generated in a Laplacian field can be interpreted in the following way. The nature of the screening effect of the Laplace equation is such that the occurrence of growth in a zone of the structure inhibits further growth in that region. In this way the Laplacian field drives the system to an asymptotic state in which the various regions of the structure are in equilibrium with each other. This perspective in the study of Laplacian growth processes has been recently proposed [5] with the introduction of a new model, the smoothed Laplacian model, in which this same mechanism of evolution is much more simple than in DLA or DBM.

The connection between this model and the Laplace equation is based on the Beurling equality [6, 7], a result of potential theory which connects the value of the electric potential in a fjord-like structure to the shape of the structure itself. Consider a fjord-like structure along the x-axis, such as that sketched in figure 1, and define $f(x)$ as the width of the cavity at x. The Beurling equation states that the solution of the Laplace equation along the x-axis decays as

$$\phi(x) \propto \exp\left\{-\pi \int_x^a \frac{\mathrm{d}z}{f(z)}\right\} \qquad x < a. \tag{1}$$

The Beurling equation in the SLM is used to define the growth probability distribution in the same way as the Laplace equation is used in the DBM: however only the global behaviour of the Laplacian field along the structure is reproduced with equation (1). In other words the effect of screening typical of the Laplace equation, which acts at all length scales in DLA and DBM (giving rise to a fractal structure) or in Saffman–Taylor fingers (producing instabilities of the finger shape to perturbations of all wavelengths [1]), is included in the SLM only at the largest scale so that the resulting cluster is compact.

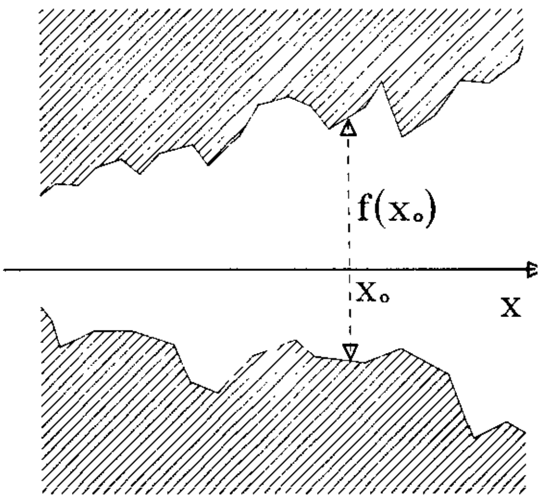

Figure 1. Schematic illustration of a section of a two dimensional fjord structure. In the electrostatic analogy, the source of the electric field is located at a distant point $x_s \gg x_0$ and the structure is supposed to be a conductor. The function $f(x)$ is the width of the fjord at any point x. The decay of the electric potential is given by equation (1).

On the other hand this same definition makes the model analytically soluble in a mean-field approximation and allows us to compare the analityical results concerning the SLM with the observed behaviour of much more complex systems such as DLA or DBM and Saffman–Taylor fingers. The asymptotic shape of SLM's fingers is obtained as the solution of an equation which describes the dynamical equilibrium of the structure in the growth probability field.

The interest of the model lies in the theoretical description of the dependence of the shape of the clusters on the strength of the Laplacian field in which they grow or on the geometry of the boundary conditions in which growth takes place.

The aim of the present paper is twofold.

(1) To compare the analytical behaviour of the SLM in strip geometry with numerical simulations of the DBM with respect to the dependence of the width of the clusters on the strength of the Laplacian field (section 2).

(2) To extend the results of the previous paper [5] to radial geometry. The basic equation in this case is solved and the results are related to recent numerical findings [4], section 3.

On the one hand we find that the theoretical description of the SLM fits fairly well the numerical results on DBM clusters; on the other hand we gain new theoretical insight in the behaviour of irreversible growth processes. Some general differences between radial and strip geometry will be discussed in the last section together with concluding remarks.

2. Smoothed Laplacian model in strip geometry

We briefly review the definition and the mean-field solution of the SLM in strip geometry; for a full treatment the interested reader is referred to [5], before turning to the numerical analysis.

The model is defined on an infinite strip of width L with periodic boundary conditions on its sides. The cluster grows along the axis of the strip and its configuration is defined, at time t, by a set of integers $\{L_k^{(r)}(t), L_k^{(l)}(t) : k = 0, 1, 2, \ldots\}$ where $L_k^{(r)}(t)$ $(L_k^{(l)}(t))$ is the number of units the cluster has grown in the direction perpendicular to the axis of the strip toward the right (left) edge in the kth row from the tip, see figure 2. The variable k is the distance from the tip and $L_{k=0}^{(r)}(t) = L_{k=0}^{(l)}(t) = 0\ \forall t$. For what follows it is more convenient to use the variables $D_k(t) = L - L_k^{(r)}(t) - L_k^{(l)}(t)$ which represent the width of the fjord left at time t by the structure at a distance k from the tip. The definition of $D_k(t)$ reflects the choice of periodic boundary conditions. These are the variables used in the definition of the growth probability which rules the evolution. Namely

$$P_k(t) = P_0(t) \exp\left\{ -c \sum_{n=1}^{k} \frac{1}{D_n(t)} \right\} \tag{2}$$

is the probability that growth occurs at time t on the right (left) side of the structure in the kth row. Hence with probability $P_k(t)$, $D_k(t+1) = D_k(t) - 1$ and $D_n(t+1) = D_n(t)\ \forall n \neq k$. With probability $P_0(t)$ (which is determined by the normalization condition) the cluster grows one unit along the axis of the strip, so that in this case $D_k(t+1) = D_{k-1}(t)\ \forall k \geqslant 1$.

c is the only parameter of the model and it modulates the strength of the screening effect in the same way as the parameter η does for DBM [3]. The similarity between this screening effect and the one typical of the Laplace equation comes from the fact that equation (2) has the same dependence of equation (1) on the shape of the structure. In view of these equations and of the definition of the DBM [3], the relation between c and the parameter η of the DBM should be $c = \pi\eta$. One has however to recall that equation (1) reproduces correctly the behaviour of the Laplacian field only in fjord-like structures, while the definition (2) of $P_k(t)$ assumes the same dependence also for the region of the tip of the cluster.

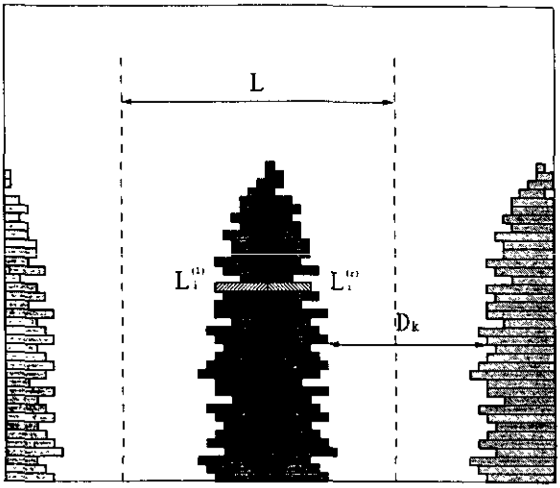

Figure 2. Smoothed Laplacian model in strip geometry. The pattern grows from the centre line of the strip, in the upper direction with probability P_0, and in the lateral one, at a distance k from the tip, with probability P_k. Periodic boundary conditions are applied to the side edges of the strip (dashed vertical lines).

The mean-field equation for $D_k(t)$ that corresponds to the above evolution rules is

$$\frac{\partial D_k(t)}{\partial t} = -P_0(t)\frac{\partial D_k(t)}{\partial k} - 2P_k(t). \tag{3}$$

Due to the ergodic property of the system for $t \to \infty$ as $\partial D_k(t)/\partial t \to 0$ and the resulting equation for D_k in this limit gives the stationary profile of the cluster. The physics contained in this equation is just the balance between growth in the two main directions, the one parallel to the strip and the one perpendicular to it. Using equation (2) for $P_k(t)$ and the rescaled variables $y = k/L$ and $d(k/L) = D_k/L$, equation (3) can be solved, with the boundary condition $d(0) = 1$, to obtain $d(y)$ in implicit form

$$y = \int_{d(y)}^{1} \frac{\mathrm{d}z}{2 + c\ln z}. \tag{4}$$

With respect to the asymptotic width of the structure it is easy to see that for $y \to \infty$ as $d(y) \to d_\infty = \exp(-2/c)$. This behaviour allows us to make some considerations. Firstly we notice that the limit $c \to 0$ is not analytical and this suggests that the same situation could occur also for DBM in the limit $\eta \to 0$. In the case of DLA, which corresponds to $c = \pi$, the relative width of the structure is known to be $1/2$. The corresponding value for the SLM is $1 - d_\infty = 1 - \exp(-2/\pi) = 0.471\ldots$ which is in close agreement. Moreover the SLM gives a clear indication of the dependence of this width on the parameter η of DBM, it should in fact behave like $\delta \propto \exp(-a/\eta)$. This suggestion has been tested performing numerical simulations of DBM aggregates in strip geometry. In view of computer limitations it has been possible to consider system sizes only up to $L = 32$. One hundred clusters were grown up to a height $H = 5\,L$ for 11 values of η from 0.5 to 3.0.

The main conceptual problem in this analysis concerns the definition of the width of DBM clusters. Two different definition have been used. The first is that given by Arneodo in [4]. The width of the structure is defined as the mid-height width of the site occupation probability distribution. The second definition is based on the behaviour of the Laplacian field along the structure. The solution of the Laplace equation decays exponentially [8] from the tip

$$\phi(h) \propto \exp\left(-\pi\frac{h}{\delta}\right)$$

where h is the distance from the tip, with a typical penetration length δ whose meaning is the size of the channel left empty by the structure. This definition has a direct physical meaning if one supposes that the universality of occupation properties in DLA, viscous fingering and Saffman–Taylor fingers recently proposed lies in the fact that in all these problems the stationary state is due to the equilibrium situation of the growing structure in the Laplacian field. Indeed even if the morphology of the structure can be very different in the three cases, the behaviour of the Laplacian field should be the same, and in particular its behaviour along the structure at the largest scale. This is why we expect that the behaviour displayed by the SLM, in which this equilibrium situation can be studied analytically, is correct also for other more complex systems of Laplacian growth.

Moreover in the first definition, in order to obtain a meaningful distribution, it is necessary to apply reflecting boundary conditions on the side edges of the cell, as done in [4]. This choice makes the system symmetric with respect to the axis of the cell. Periodic boundary conditions instead do not indentify an axis of symmetry so that the distribution of occupation turns out to be flat and the corresponding width remains undefined. This does not happen if one defines the width of the clusters using the penetration length of the Laplace equation that is well defined also for periodic boundary conditions. For the SLM the choice of boundary conditions enters in the relation between D_k, $L_k^{(r)}$ and $L_k^{(l)}$. This definition can be modified for the case of reflecting boundary conditions but the final result for the shape of the cluster is the same. The choice of reflecting boundary conditions, in the end, is irrelevant.

The fjord width δ obtained by both methods was fitted to a functional form of the following kind

$$\delta = b\exp(-a/\eta). \qquad (6)$$

The values of the coefficients for $L = 16$ and $L = 32$ are given in table 1. In figure 3 the fit is shown for $L = 32$ where the error bars represent the penetration length data while the squares are obtained by the mid-height width of the occupation probability profile.

In particular the plot of the fit shows that for $\eta = 1$ the width obtained from the penetration length is very close to $L/2$, while the mid-height width of the occupation distribution differs from $L/2$ in a sensible way. In order to test the correctness of the simulations, 5 clusters were grown with $L = 64$ and $\eta = 1$ and the distribution of occupation was computed. The width at mid-height is in this case very close to $L/2$ and the form of the distribution itself coincides with the one found by Arnéodo *et al* [4] for the same system size. This suggests that the distribution of occupation does not give a reliable definition of the cluster width for the system sizes analysed

Table 1. Values of the coefficients of the fit shown in figure 3 for the width of the fjiord left by DBM clusters. The coefficients refer to the parameters in equation (6). The subscript ϕ labels the data obtained from the decay of the solution of the Laplace equation along the clusters, while p labels the ones obtained from the profile of the occupation distribution. The nonlinearity of the fit makes the evaluation of statistical errors problematic. For this reason they are not reported in the table. The last column reports for comparison the theoretical value of a that is $a_{\text{theor}} = 2/\pi$.

L	b_ϕ	b_{p}	a_ϕ	a_{p}	a_{theor}
16	16.5	14.6	0.699	0.996	0.637
32	36.2	27.6	0.801	0.833	0.673

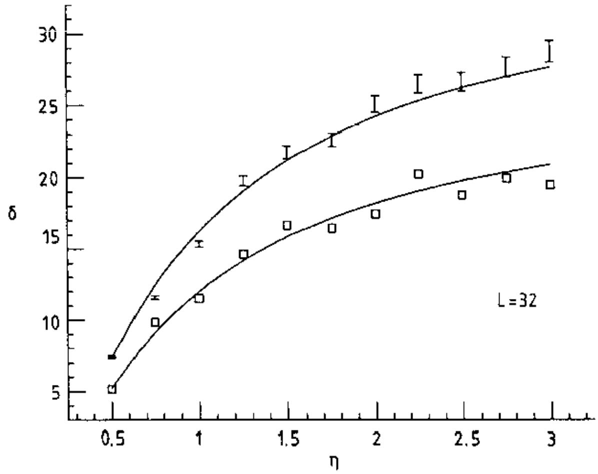

Figure 3. Dependence on the parameter η of the width of the fjord left by DBM clusters, grown in a strip of size $L = 32$. The error bars refer to the penetration length data, while the squares are obtained from the occupation profile.

and that the relation between this width and the one obtained from the decay of the Laplacian field is not trivial.

The values of the coefficients of the fit disagree from the ones suggested by the SLM, $a = 2/\pi$ and $b = L$, in a sensible way and are not constant with respect to L. This may be a consequence of the smallness of the system sizes analysed or of the approximations contained in the SLM, specially with respect to the description of the growth process in the tip region. Larger simulations are necessary to give conclusive answers in this respect. The key point is however that the functional dependence of the width of the cluster on the strength of the Laplacian field indicated by the SLM is reproduced fairly well by DBM clusters for both values of L.

3. Smoothed Laplacian model in radial geometry

In the radial case the clusters are defined in sector shaped cells (see figure 4) of vertex angle ϑ and grow along the axis of the cell, from the vertex outwards in the radial direction, and from the axis towards the two edges, in the tangential one. Again periodic boundary conditions are applied to the side edges of the cell.

At a certain time t the distance of the tip of the cluster from the vertex is an integer $H(t)$ and the configuration can be described by a set of integers $\{L_k^{(r)}(t), L_k^{(l)}(t) : k = 1, \ldots, H(t)\}$ with $L^{(r)}_{k=H(t)}(t) = L^{(l)}_{k=H(t)}(t) = 0\ \forall t$. These variables, as for the linear case, represent the amount of growth at a distance k from the vertex. In the radial case however two possible choices are available for these variables, growth can occur along the direction perpendicular to the axis of the cell or along the arc of circumference. The second choice appears to be the most natural for this geometry even if it makes it difficult to define the model on a lattice. This will not be necessary in the present paper so that the second definition will be used. Note that in the present case the meaning of the variable k is different from the one it had in the previous section and that the number of values of k necessary for the theoretical description of the process increases in time.

Defining the variable $D_k(t) = \vartheta k - L_k^{(r)} - L_k^{(l)}$ in a similar way as in the previous section, assuming periodic boundary conditions on the sides of the cell, at time t growth will occur on the kth arc of circumference from the centre with probability

$$P_k(t) = P_0(t) \exp\left\{-c \sum_{n=k}^{H(t)} \frac{1}{D_n(t)}\right\} \tag{7}$$

and the corresponding variable $L_k^{(r)}(t)$ (or $L_k^{(l)}(t)$) will be increased by unity, while $D_k(t)$ will decrease by one unit. With probability P_0, that is again determined by the normalization condition, growth will occur on the tip of the structure so that $H(t+1) = H(t) + 1$. The mean evolution can be summarized by the following set of equations

$$\begin{cases} \dfrac{\partial D_k(t)}{\partial t} = -2P_k(t) & k < H(t) \\ D_k(t) = \vartheta k & k \geqslant H(t) \end{cases}$$
$$\begin{cases} \dfrac{\partial H(t)}{\partial t} = P_0(t) \\ H(t=0) = 0\,. \end{cases} \tag{8}$$

The argument that has led to equations (8) is not as rigorous and systematic as that which gives the corresponding equations for the linear case [5]. The physical content is however the same: the mean-field approximation for the evolution of a structure in a stochastic-field whose functional dependence on the shape of the structure itself is that of the Beurling equality. The relevant length scale in the problem is clearly $H(t)$ and it is useful to introduce the rescaled variables $d(r)$ and r defined as

$$\begin{cases} d(r) = \dfrac{D_k(t)}{H(t)} \\ r = \dfrac{k}{H(t)}\,. \end{cases}$$

These variables indeed contain the information about the shape of the structure while $H(t)$ contains the dependence of the size of the structure on time. Indeed it

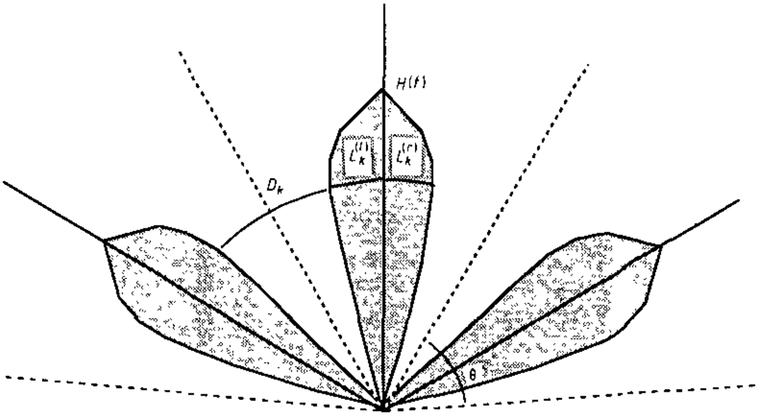

Figure 4. Definition of the SLM in radial geometry. The lengths $L_k^{(r)}(t)$, $L_k^{(l)}(t)$ and $D_k(t)$ are measured along the arc of circle at distance k from the origin. Three identical fingers are shown in the figure to represent the choice of periodic boundary conditions.

can be easily verified that in terms of the function $d(r)$ the first of equations (8), using equation (7), gives an equation for the mean shape $d(r)$ of the cluster that is completely independent of the time variable

$$\begin{cases} r\dot{d}(r) = d(r) + 2\exp\left(-c\int_r^1 \frac{\mathrm{d}z}{d(z)}\right) \\ d(1) = \vartheta. \end{cases} \tag{10}$$

The evolution is instead described by the second of equations (8) that in its turn depend on the shape of the cluster through the normalization condition. Using equation (10) integrated from 0 to 1 on r, this dependence can be made more explicit

$$\begin{aligned} \frac{\mathrm{d}H(t)}{\mathrm{d}t} = P_0(t) &= \left[1 + 2\sum_{k=1}^{H(t)} \exp\left(-c\sum_{n=k}^{H(t)} 1/D_n(t)\right)\right]^{-1} \\ &= \left[1 + 2H\int_0^1 \exp\left(-c\int_r^1 \mathrm{d}z/d(z)\right)\mathrm{d}r\right]^{-1} \\ &= \left[1 + H\left(\vartheta - 2\int_0^1 d(r)\,\mathrm{d}r\right)\right]^{-1}. \end{aligned} \tag{11}$$

This equation can be easily solved for $H(t)$. Note only that the shape does not affect the functional dependence of $H(t)$ on t. $H(t)$ is indeed proportional to the square root of t (that is just a consequence of the fact that the cluster is compact and its mass-length dimension is 2). The form of $d(r)$ only affects the prefactor. Let us now focus on equation (10) and look for a solution. It is useful to make the following change of variables:

$$\begin{cases} \varphi(\xi) = d(r)/r \\ \xi = \ln r. \end{cases} \tag{12}$$

φ has the same meaning of the angle in a polar coordinate system. Equation (10) now reads

$$e^{\xi}\dot{\varphi}(\xi) = 2\exp\left(-c\int_{\xi}^{0}\frac{d\eta}{\varphi(\eta)}\right). \tag{13}$$

The factor e^{ξ} in the left-hand side can be brought to the other side and then into the integral at the exponent thus giving

$$\dot{\varphi}(\xi) = 2\exp\left[\int_{\xi}^{0}\left(1-\frac{c}{\varphi(\eta)}\right)d\eta\right] \tag{14}$$

with the boundary condition $\varphi(0) = \vartheta$. After taking the derivative of both terms, one gets a second-order nonlinear equation

$$\begin{cases} \ddot{\varphi}(\xi) = -\dot{\varphi}(\xi)\left(1-\dfrac{c}{\varphi(\xi)}\right) \\ \dot{\varphi}(0) = 2 \\ \varphi(0) = \vartheta \end{cases} \tag{15}$$

where the boundary condition on the derivative comes from equation (14) evaluated in $\xi = 0$. This equation can be solved with two successive integrations to obtain $\varphi(\xi)$ in implicit form

$$\xi = -\int_{\varphi(\xi)}^{\vartheta}\frac{d\psi}{2+\vartheta-c\ln\vartheta-\psi+c\ln\psi}. \tag{16}$$

The solution is shown in figure 5, in terms of the variable $l(r) = (\vartheta r - r\varphi(r))/2$ that coincides with the rescaled variable relative to $L_k^{(l)}(t)$ and is more closely related to the actual shape of the finger, even if one has to remember that the variable $l(r)$ is measured on the arc of circumference. The finger width grows from zero, at the tip, to a maximum value. It can be shown that this maximum is obtained for $\varphi_{max} = \vartheta\exp(-2/c)$ and that the corresponding value r_{max} moves closer to one for lower ϑ and c values. Further decreasing r, $l(r)$ decreases from the maximum to zero approaching a linear behaviour. From equation (16) it is easy to see that for $\xi \to -\infty$ (that is $r \to 0$) $\varphi(\xi)$ tends to a value φ_0 that is the solution of the following equation

$$2+\vartheta-c\ln\vartheta-\varphi_0+c\ln\varphi_0 = 0. \tag{17}$$

The dependence of φ_0 on ϑ and c can be understood with the help of the construction shown in figure 6. The curve $y = x - c\ln x$ has a minimum for $x = c$. For a given value of ϑ, φ_0 is given by the value of x in which the line $y = 2+\vartheta-c\ln\vartheta$ intersects the curve $y = x - c\ln x$. Since $\varphi_0 < \vartheta$ only the intersection on the left has to be considered. This means that φ_0 is always smaller than c and it increases for $\vartheta < c$ and decreases for $\vartheta > c$. This behaviour is confirmed by a crude estimation of φ_0, which can be easily obtained from equation (17)

$$\vartheta\exp\left(-\frac{2+\vartheta}{c}\right) < \varphi_0 < \vartheta\exp\left(1-\frac{2+\vartheta}{c}\right). \tag{18}$$

The physical meaning of the angle φ_0 is the vertex angle of the cone that will be left asymptotically unoccupied by the structure. The quantity $\lambda = (\vartheta - \varphi_0)/\vartheta$ instead is the fraction of the arc of circumference occupied asymptotically by the cluster. This quantity has recently been calculated performing numerical simulation of DLA in sector shaped cells [4] that has revealed a linear dependence of λ on ϑ, equation 1 of [4]. The corresponding behaviour in the SLM is somehow more complex and can be expressed, using equation (17), in implicit form

$$\vartheta = -\frac{2 + c\ln(1-\lambda)}{\lambda}. \tag{19}$$

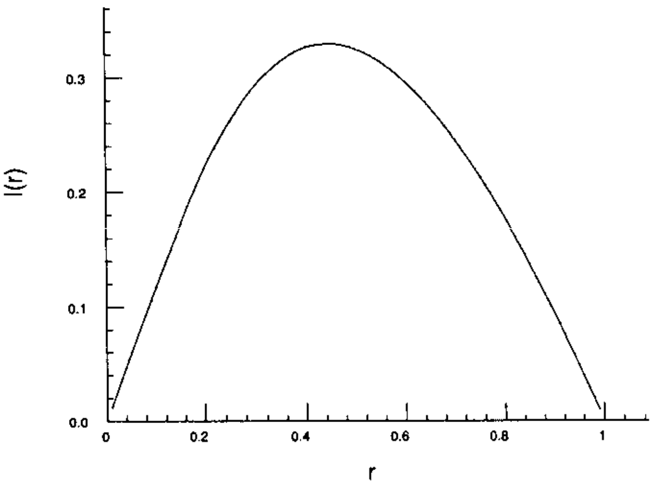

Figure 5. Solution of equation (14) for the rescaled variable $l(r) = r(\vartheta - \varphi(r))/2$ which corresponds to $L_k^{(l)}(t)$, in the case $c = \pi$ and $\vartheta = \pi/2$.

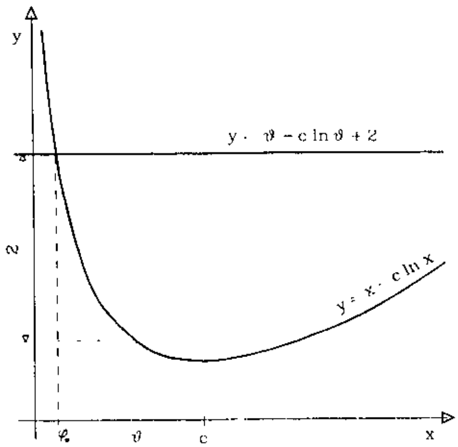

Figure 6. Graphic solution of equation (15) for φ_0 as a function of c and ϑ.

Note first of all that for $\vartheta \to 0$, λ tends to the value $1 - \exp(-2/c)$ which is consistent both with the result of the SLM in strip geometry and with the numerical result of Arneodo *et al* [4] (i.e. $\lambda(\vartheta = 0) = 0.5$) in the case of DLA ($c = \pi$). Moreover the behaviour of equation (19) for $c = \pi$ in the range $0 < \vartheta < \pi/2$, is almost indistinguishable form a straight line (see figure 7) so that this behaviour is consistent with the numerical result [4]. However the coefficient of a linear fit of equation (19) for $c = \pi$ (that, evaluating $\mathrm{d}\lambda/\mathrm{d}\vartheta$ in $\vartheta = 0$, turns out to be approximatively $1.56 \times 10^{-3}\ \mathrm{rad}^{-1}$) is smaller than one half of the one found numerically by Arneodo *et al* ($(3.4 \pm 0.2) \times 10^{-3}\ \mathrm{rad}^{-1}$). The reason of this discrepancy lies in the fact that the description provided by the SLM of the growth process in the tip region is very different from that of DLA. The tip of SLM clusters is a sharp cone whose vertex angle is $\pi/2$ for all ϑ and c values†, while the tip's shape of Saffman–Taylor fingers is known to be smooth. The situation clearly gets worse for larger ϑ values and this is consistent with a linear divergence of $\lambda(\vartheta)$ in the two cases. An interesting suggestion of equation (19) is that the linear behaviour of λ on ϑ is not exact. Greater deviation from a linear behaviour can be observed in equation (19) for small c values, so that this point could be checked performing numerical simulations on DBM with small values of η.

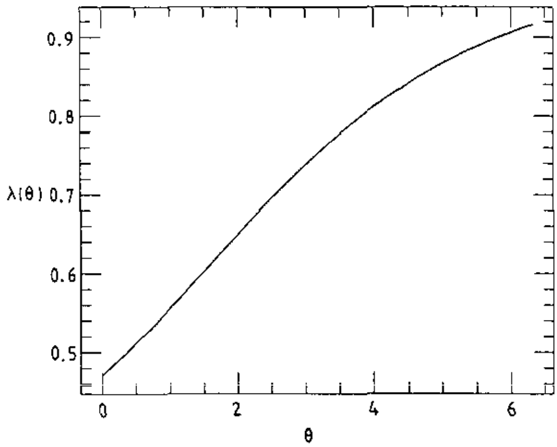

Figure 7. Plot of equation (19) for $\lambda(\vartheta)$ in the case $c = \pi$. For larger values of ϑ the deviation from a linear behaviour becomes more pronounced and λ tends asymptotically to 1. However the physical range of ϑ increases only to 2π.

Finally it is interesting to investigate the behaviour of the SLM in radial geometry near the vertex of the cell. Firstly we analyse the way in which the solution $d(r)$ approaches the linear behaviour for $r \ll 1$. Using the variable $\epsilon(r) = \varphi(r) - \varphi_0$, one finds from equation (16) an expression for $\mathrm{d}\epsilon/\mathrm{d}\xi$

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}\xi} = \epsilon - c \ln\left(1 + \frac{\epsilon}{\varphi_0}\right) \simeq \left(1 - \frac{c}{\varphi_0}\right)\epsilon + \mathrm{O}\left(\epsilon^2\right) \tag{20}$$

that to linear order in ϵ, gives after few passages

$$d(r) = r(\varphi_0 + \epsilon(r)) \simeq \varphi_0 r + A r^{c/\varphi_0}. \tag{21}$$

† This is due to the fact that the boundary condition on the derivative $\dot{d}(r)$ in $r = 1$ (or of $\dot{\varphi}(\xi)$ in $\xi = 0$) does not depend on c and ϑ.

Considering the dependence of φ_0 on c, equation (18), we conclude that the convergence to a linear behaviour is faster for lower c values. The ratio c/φ_0 enters also in the power law behaviour of the growth probability near the vertex. It can be easily shown, using equation (21), that for for $k \ll H(t)$

$$P_k(t) \simeq P_0(t) B \left(\frac{k}{H(t)}\right)^{c/\varphi_0} \tag{22}$$

and, considering that $P_0(t) = 1/H(t)$ and $H(t) \propto \sqrt{t}$, we get

$$P_k(t) \simeq B' t^{-[(c/\varphi_0+1)/2]}. \tag{23}$$

The last equation gives the behaviour of the probability at a distance k in the limit $t \to \infty$ for any k. The fact that ϑ is always smaller than c makes this probability decay faster then $1/t$ and this is a necessary condition for the freezing of the structure†. The ratio c/φ_0 then is relevant because it makes this freezing more, or less, rapid in time.

These results show that the asymptotic part of the structure extends over a fraction of the total radius that is larger for smaller c values and that the freezing process is faster the smaller c is. The same situation is likely to occur in DBM, and this is somewhat surprising if one thinks that in this limit ($c \to 0$, $\eta \to 0$) the strength of the Laplacian screening effect decreases.

In summary the analysis of the SLM in radial geometry displays a behaviour much more complex than the one obtained in strip geometry. This is primairily due to the inclusion of another external parameter, the vertex angle ϑ of the cell. Moreover the same geometry of the problem makes space and time variables dependent. Introducing opportunely rescaled variables however the problem of the time evolution of the cluster size and of his geometric shape can be decoupled and solved. Interesting correspondences can be found with recent numerical analysis on DLA [4]. Moreover the SLM provides further suggestions about the theoretical behaviour of Laplacian growth processes. It is however necessary to remind that the theoretical analysis for the radial case of the SLM is much less systematic and rigorous than that of the linear case [5]. In particular the validity of the mean-field approximation in radial geometry has not been proved so far with the study of fluctuations and of the dynamical stability of the mean-field solution.

4. Discussion

In the previous sections more attention has been focused on the relation between the analytical results on the SLM and the behaviour of DLA and DBM, than on the results themselves. This attitude reflects the main motivation for the introduction of the model, that is to provide theoretical information on the behaviour of such very complex models, in a simple way. It is however necessary to recall that the above-mentioned relation is far from being rigorous and this is the reason why the relevance of the description provided by the SLM for the behaviour of the DBM lies more in the functional dependence on the various parameters than on their numerical values.

† In view of the Borel Cantelli lemma [9] this condition ensures that only a finite number of growth events occur at a distance k from the vertex.

The inadequacy of numerical values is mainly due to the fact that the description of the growth process in the tip region is inappropriate for Laplacian growth processes. The definition of the model given in the previous sections is the simplest one and can be easily improved to take into account of this problem.

On the other hand the scenario outlined for the SLM in this paper allows us to make a comparison between the radial and the strip geometries, with respect to the theoretical description of the growth process, whose validity is more general than the specific case of the SLM. The main difference between the two geometries of boundary conditions can be expressed, loosely speaking, in terms of volume of phase space. In strip geometry the volume of phase space is constant in time while in the radial case the phase space gets progressively larger. This is evident from the scaling form of the shape of the cluster which involves the size of the structure that, in the linear case, is independent of time, while in the radial one it increases with time.

The asymptotic state of the system in strip geometry is that of an ergodic stationary process that possesses stability properties and well behaved fluctuations. In radial geometry the process is not stationary and the asymptotic properties come as a result of a problem of 'equilibrium' which is intrinsically irreversible. It is then likely that the system, in this geometry, is much more complex and sensitive to the details of the model.

The behaviour of the DLA is an example of this situation. In radial geometry (that would coincide with a sector shaped cell of angle $\vartheta = 2\pi$) the indications of numerical simulations lead to a rather unclear picture. The system, that is initially isotropic, undergoes a transition to a structure characterized by few main arms. The mechanism of selection of the number of arms is influenced drastically by the geometry of the underlying lattice [10]. In its turn, according to an analytical result due to Ball [11], the number of arms depends on the mass-length dimension of the cluster. Moreover it seems that, for very large sizes, the mass-length dimension of the cluster decreases and the system becomes deterministic [10].

In strip geometry the growth direction is uniquely defined and the geometry of the lattice does not influence the asymptotic properties. After the early stages of growth, in which the scaling properties of the structure are not totally understood [12], the system enters a steady state of growth in which it can be described as a Markovian process.

The crucial point is that the 'thermodynamic' limit $L \to \infty$ is well defined for such a stable state and the system possesses well defined properties in this limit. In the case of the SLM these properties concern the shape of the structure, in the sense that, in strip geometry, typical configurations follow the mean-field profile, equation (4), and fluctuations vanish in this limit. What are the analogous properties in the case of DLA and DBM? Are typical configurations in the limit $L \to \infty$ characterized by a well defined value of the fractal dimension? What is the limiting profile of the site occupation distribution? The stationary nature of the process in strip geometry allows us to address such unambiguous questions, while the same problems in radial geometry require a deeper understanding of the nature of the asymptotic state.

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